Journal of Basic and Applied Engineering Research
p-ISSN: 2350-0077; e-ISSN: 2350-0255; Volume 6, Issue 7; July-September, 2019, pp. 382-384

http://www.krishisanskriti.org/Publication.html

MORE IDENTITIES OF ROGERS-RAMANUJAN TYPE

Shaikh Fokor Uddin Ali Ahmed

Department of Mathematics, F. A. Ahmed College,
Garoimari, Kamrup Assam, India
E mail: fokoruddin86@gmail.com

Abstract

The following two Identities, namely, for $|q|<1$,

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n\not\equiv 0,\pm 2 \ (mod\ 5)$$

and

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\prod_{n=1}^{\infty}\frac{1}{1-q^n}, n\not\equiv 0,\pm 1 \ (mod\ 5)$$

where

$(q;q)_n=\prod_{j=1}^{n}(1-q^j)$ & $(q;q)_\infty=\prod_{j=1}^{\infty}(1-q^j)$

and

$(a_1, a_2, \ldots a_s; q)_\infty = (a_1;q)_\infty\, (a_2;q)_\infty \ldots\ldots (a_s;q)_\infty$

are known as celebrated original Rogers-Ramanujan Identity. These two identities have motivated extensive research over the past hundred years, The Rogers-Ramanujan Identities has two aspects: one analytical and the other is combinatorial.

The present paper intends to give a brief discussion on Original Rogers-Ramanujan Identities and to derive some more identities of Rogers-Ramanujan Type related to modulo 5, 8 and 12 analytically by using some general transformation between Basic Hypergeometric Series with the incorporation of some identities from Lucy Slater's famous list of 130 identities of Rogers-Ramanujan type.

Keywords: *Rogers-Ramanujan Identity, Slater's Identity, Basic Hypergeometric Series, Jacobi's Triple Product Identity, Bailey Pair etc.*

Introduction:

For |q|<1, the q-shifted factorial is defined by $(a;q)_0 = 1$

$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$, for n≥1

and $(a;q)_\infty = \prod_{k=1}^{\infty}(1-aq^k)$.

It follows that $(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}$

The multiple q-shifted factorial is defined by

$$(a_1,a_2,\ldots,a_m;q)_n=(a_1;q)_n(a_2;q)_n\ldots(a_m;q)_n$$

$$(a_1,a_2,\ldots,a_m;q)_\infty=(a_1;q)_\infty(a_2;q)_\infty\ldots(a_m;q)_\infty$$

The Basic Hyper geometric Series is

$${}_{p+1}\phi_{p+r}\left(\begin{matrix} a_1,a_2,\ldots\ldots,a_{p+1};q;x \\ b_1,b_2,\ldots\ldots,b_{p+r}\end{matrix}\right)$$

$$=\sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\ldots(a_{p+1};q)_n x^n(-1)^{nr}q^{\frac{n(n-1)r}{2}}}{(q;q)_n(b_1:q)_n(b_2;q)_n\ldots(b_{p+r};q)_n}$$

The series ${}_{p+1}\phi_{p+r}$ converges for all positive integers r and for all x. For r=0 it converges only when |x|<1.

Ramanujan's Theta function: Ramanujan's Theta function ([4], P.11, Eq. (1.1.5)) is defined as

$f(a, b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}$, for |ab|<1.

The following special cases of f (a, b) arise so often that they were given their own notation by Ramanujan ([4], P.11) as follows:

$\varphi(q) = f(q, q)$

$\psi(q) = f(q, q^3)$

$f(-q) = f(-q, -q^2)$

Jacobi's triple product identity :([3], P.2, Eq. (1.1.7))

For |ab| < 1, $f(a, b) = (-a, -b, ab; ab)_\infty$

An immediate corollary ([3], P-2, Eq. (1.1.8), (1.1.9), (1.1.10)) of this identity is thus

$f(-q) = (q;q)_\infty$

$\varphi(q) = \frac{(q;q)_\infty}{(-q;q)_\infty}$

$\psi(q) = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}$

We now list some general transformations. Most of them can be derived as limiting case of transformations between basic hyper geometric series. Let a, b, c, d, γ and q ∈**C**, |q|<1. Then

$$\sum_{n=0}^{\infty}\frac{q^{n^2}\gamma^n}{(q/b;q)_n(q;q)_n}$$

$$=(-\gamma q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}\gamma^n(-\frac{q}{b};q)_{2n}}{(q^2;q^2)_n(\frac{q^2}{b^2};q^2)_n(-\gamma q^2;q^2)_n} \quad (1.1)$$

$$\sum_{n=0}^{\infty}\frac{(d;q)_{2n}q^{n^2-n}(-\frac{c^2}{d^2})^n}{(q^2;q^2)_n(c;q)_{2n}}$$

$$=\frac{(\frac{c^2}{d^2};q^2)_\infty}{(c;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n}\gamma^n(-c)^n}{(q;q)_n(-\frac{c}{d};q)_n} \quad (1.2)$$

$$\sum_{n=0}^{\infty}\frac{q^{3n^2-2n}(-a^2)^n}{(q^2;q^2)_n(a;q)_{2n}}=\frac{1}{(a;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2-n}(-a)^n}{(q;q)_n} \quad (1.3)$$

$$\sum_{n=0}^{\infty}\frac{(a;q)_nq^{n^2-n}(-b)^n}{(q;q)_n(ab;q^2)_n}=\frac{(b;q^2)_\infty}{(ab;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(a;q^2)_nq^{n^2-n}(-bq)^n}{(q^2;q^2)_n(b;q^2)_n} \quad (1.4)$$

and

$$\sum_{n=0}^{\infty}\frac{(a^2;q)_nq^{(3n^2+n)/2}(a)^{2n}}{(q;q)_n}=$$

$$\frac{(a^2q;q)_\infty}{(-aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-a;q)_nq^{(n^2-n)/2}(aq)^n}{(aq,q;q)_n} \quad (1.5)$$

The transformation (1.1) is appeared as (6.1.12) on page 41 in [3]. The transformation (1.2) follows from (3.5.4) on pages 77-78 in [5], after replacing c with aq/c, then letting $a\to 0$ and finally letting$b\to\infty$. It is also appeared as (6.1.17) on page 41 in [3]. The transformation (1.3) follows from (1.2) upon letting$d\to\infty$, and then replacing c *with* a. This transformation is also appeared as (6.1.18) on page 41 in [3]. The transformation (1.4) follows from a result of Andrews in [6] (see also Corollary 1.2.3 of [7], where it follows after replacing t by t/b , then letting $b\to\infty$ and finally replacing t by b). Finally, the transformation (1.5) is appeared as (6.1.21) on page 42 in [3].

2. We shall now introduce some identities from the Lucy Slater's famous list of Rogers-Ramanujan Type Identities. Each of them below that appears in [3] is designated with a "Slater number" S.n.

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\frac{f(-q,-q^4)}{f(-q)}, \quad (2.1)$$

([3], Equation (2.5.1) p.11); ($S_{.14}$)

$$\sum_{n=0}^{\infty}\frac{(-1)^nq^{3n^2}}{(-q;q^2)_n(q^4;q^4)_n}=\frac{f(-q^2,-q^3)}{f(-q^2)}, \quad (2.2)$$

([3], Equation (2.5.7) p.12); ($S_{.19}$)

$$\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(q;q)_{2n+1}}=\frac{f(-q^2,-q^{10})}{f(-q)} \quad (2.3)$$

([3], Equation (2.12.2) p.17); ($S_{.50}$)

$$\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q)_{2n+1}}=\frac{f(q,q^7)}{f(-q^2)} \quad (2.4)$$

([3], Equation (2.8.9) p.15); ($S_{.38}$)

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q;q)_{2n}}=\frac{f(q^3,q^5)}{f(-q^2)} \quad (2.5)$$

([3], Equation (2.8.10) p.15); ($S_{.39}$)

$$\sum_{n=0}^{\infty}\frac{q^{n(n+2)}(-q;q^2)_n}{(q^4;q^4)_n}=\frac{f(-q,-q^5)}{\psi(-q)} \quad (2.6)$$

([3], Equation (2.6.2) p.13)

3. Identities related to modulo 8:

Replacing q by q^2 in (1.1), we get

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}\gamma^n}{(q^2/b;q^2)_n(q^2;q^2)_n}$$

$$=(-\gamma q^4;q^4)_\infty\sum_{n=0}^{\infty}\frac{q^{2n^2}\gamma^n(-q^2/b;q^2)_{2n}}{(q^4;q^4)_n(\frac{q^4}{b^2};q^4)_n(-\gamma q^4;q^4)_n} \quad (3.1)$$

Setting $b=1/q$ and $\gamma=q^2$ in (3.1), we have

$$(-q^6;q^4)_\infty\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q^3;q^2)_{2n}}{(q^4;q^4)_n(q^6;q^4)_n(-q^6;q^4)_n}$$

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q^3;q^2)_n(q^2;q^2)_n}$$

which on some reduction, yields

$$\frac{(-q^6;q^4)_\infty}{(1-q)}\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q^3;q^2)_{2n}}{(q^4;q^2)_{2n}(-q^6;q^4)_n}=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{(q;q)_{2n+1}} \quad (3.2)$$

Now using (2.4) in (3.2) we get the following identity

$$\frac{(-q^6;q^4)_\infty}{(1-q)}\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}(-q^3;q^2)_{2n}}{(q^4;q^2)_{2n}(-q^6;q^4)_n}=\frac{f(q,q^7)}{f(-q^2)}=\frac{(-q,-q^7,q^8;q^8)_\infty}{(q^2;q^2)_\infty}$$

$$=\frac{(-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty}{(-q;q)_\infty(q;q)_\infty}=\prod_{n=1}^{\infty}\frac{1}{1+q^n}\cdot\prod_{m=1}^{\infty}\frac{1}{1-q^m} \quad (3.3)$$

where n ≢ 1,7 (mod 8) & m ≢ 0 (mod 8)

Again, placing $q^{1/2}$ in place of q in transformation (1.1), we have

$$\sum_{n=0}^{\infty}\frac{q^{n^2/2}\gamma^n}{(q^{1/2}/b;q^{1/2})_n(q^{1/2};q^{1/2})_n}$$

$$=(-\gamma q;q)_\infty\sum_{n=0}^{\infty}\frac{q^{\frac{n^2}{2}}\gamma^n(-\frac{q^{\frac{1}{2}}}{b};q^{\frac{1}{2}})_{2n}}{(q;q)_n(\frac{q}{b^2};q)_n(-\gamma q;q)_n} \quad (3.4)$$

which for $b=q^{1/4}$ and $\gamma=1$ gives

$$\sum_{n=0}^{\infty}\frac{q^{n^2/2}}{(q^{1/4};q^{1/2})_n(q^{1/2};q^{1/2})_n}$$

$$=(-q;q)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2/2}(-q^{1/4};q^{1/2})_{2n}}{(q;q)_n(q^{7/8};q)_n(-q;q)_n}$$

Now, taking $q \rightarrow q^4$ we get

$$(-q^4;q^4)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2}(-q;q^2)_{2n}}{(q^8;q^8)_n\,(q^{7/2};q^4)_n}$$

$$= \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q^2)_n(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}}$$

$$= \frac{f(q^3,q^5)}{f(-q^2)}, \text{ (by 2.5)}$$

$$= \frac{(-q^3,-q^5,q^8;q^8)_\infty}{(q^2;q^2)_\infty}$$

$$= \frac{(-q^3;q^8)_\infty(-q^5;q^8)_\infty(q^8;q^8)_\infty}{(-q;q)_\infty(q;q)_\infty}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1+q^n} \cdot \prod_{m=1}^{\infty} \frac{1}{1-q^m} \tag{3.5}$$

where n ≢ 3,5 (mod 8) & m ≢ 0 (mod 8)

4. Identities related to modulo 5:

Replacing q by $q^{1/2}$ in (1.3), we get

$$\sum_{n=0}^{\infty} \frac{q^{(3n^2-2n)/2}(-a^2)^n}{(q;q)_n(a;q^{1/2})_{2n}}$$

$$= \frac{1}{(a;q^{1/2})_\infty} \sum_{n=0}^{\infty} \frac{q^{(n^2-n)/2}(-a)^n}{(q^{1/2};q^{1/2})_n} \tag{4.1}$$

Setting $a = -q^{1/2}$ in (4.1), we have, on some simplification, the following:

$$\frac{1}{(-q^{1/2};q^{1/2})_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q^{1/2};q^{1/2})_n} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2/2}}{(-q^{1/2};q)_n(q^2;q^2)_n} \tag{4.2}$$

Now taking $q \rightarrow q^2$ in (4.2) and then using (2.2), we obtain the following identity:

$$\frac{1}{(-q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(qq)_n} = \frac{f(-q^2,-q^3)}{f(-q^2)}$$

$$= \frac{(q^2,q^3,q^5;q^5)_\infty}{(q^2;q^2)_\infty} = \frac{(q^2,q^3,q^5;q^5)_\infty}{(-q;q)_\infty(q;q)_\infty}$$

Hence it reduces to the original Rogers-Ramanujan Identity, viz,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(qq)_n} = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q;q)_\infty} = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \text{ n} \not\equiv 0, 2, 3 \text{ (mod 5)} \tag{4.3}$$

Again, Setting $a = -q$ in (4.1), we have

$$\sum_{n=0}^{\infty} \frac{q^{(3n^2+2n)/2}}{(q;q)_n(-q;q^{1/2})_{2n}} = \frac{1}{(-q;q^{1/2})_\infty} \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2}}{(q^{1/2};q^{1/2})_n} \tag{4.4}$$

Now using (2.1) in (4.4) after replacing q by q^2, it yields the following identity

$$(-q^2;q)_\infty \sum_{n=0}^{\infty} \frac{q^{(3n^2+2n)}}{(q^2;q^2)_n(-q^2;q)_{2n}} = \frac{f(-q,-q^4)}{f(-q)}$$

$$= \frac{(q,q^4,q^5;q^5)_\infty}{(q;q)_\infty} = \frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^5;q^5)_\infty}{(q;q)_\infty}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \text{ n} \not\equiv 0, 1, 4 \text{ (mod 5)} \tag{4.5}$$

5. Identities related to modulo12:

Placing q^2 in place of q in transformation (1.4), we have

$$\sum_{n=0}^{\infty} \frac{(a^2;q^2)_n q^{(3n^2+n)}(a)^{2n}}{(q^2;q^2)_n}$$

$$= \frac{(a^2q^2;q^2)_\infty}{(-aq^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-a;q^2)_n q^{n^2-n}(aq^2)^n}{(aq^2,q^2;q^2)_n} \tag{5.1}$$

Setting a=q in (5.1), it reduces to:

$$\sum_{n=0}^{\infty} \frac{(q^2;q^2)_n q^{(3n^2+3n)}}{(q^2;q^2)_n} = \frac{(q^4;q^2)_\infty}{(-q^3;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^3;q^2)_n(q^2;q^2)_n}$$

which reduces to the following identity after some reduction:

$$\frac{(-q^3;q^2)_\infty}{(q^4;q^2)_\infty(1-q)} \sum_{n=0}^{\infty} \frac{(q^2;q^2)_n q^{(3n^2+3n)}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q;q)_{2n+1}}$$

$$= \frac{f(-q^2,-q^{10})}{f(-q)}, \text{ (on using (2.3))}$$

$$= \frac{(q^2,q^{10},q^{12};q^{12})_\infty}{(q;q)_\infty} = \frac{(q^2;q^{12})_\infty(q^{10};q^{12})_\infty(q^{12};q^{12})_\infty}{(q;q)_\infty}$$

$$= \prod_{n=1}^{\infty} \frac{1}{1-q^n}, \text{ n} \not\equiv 0, 2, 10 \text{ (mod 12)} \tag{5.2}$$

Acknowledgment:

The author thanks Dr. T. K. Datta, Retd. Professor, Gauhati University and late Dr. P. Rajkhowa, passed sr. Associate Professor, G.U. for their valuable suggestions, discussions and encouragements.

References

[1] F.H. Jackson, Transformation of q-series, Messenger of Math 39 (1910) 145-153.

[2] G.E. Andrews,On Basic Hypergeometric Series, mock theta functions, and partitions I, Quart. J. Math. Oxford Ser. (2) 17 (1966) 64-80.

[3] J. M. Laughlin, A. V. Sills & P. Zimmer, The Electronic Journal of Combinatorics 15(2008), #DS15.

[4] G. E. Andrews and B. C. Berndt, Ramanujan Lost Note Book, part 1, Springer 2005.

[5] G. Gasper and M. Rahman, Basic Hypergeometric Series, 2nd ed., Cambridge, 2004.

[6] G.E. Andrews, On Basic Hypergeometric Series, mock theta functions, and partitions II, Quart. J. Math. Oxford Ser. (2) 17 (1966) 132-143.

[7] G. E. Andrews and B. C. Berndt, Ramanujan Lost Note Book, part I, Springer. 2005

[8] G. E. Andrews and B. C. Berndt, Ramanujan Lost Note Book, part II, Springer.

[9] L. J. Slater, Further Identities of Rogers-Ramanujan Type. Proc. London Math. Soc.54 (1952) 147-167.